International Journal of Theoretical Physics, Vol. 23, No. 7, 1984

Introduction to Majorana Masses

Philip D. Mannheim

Department of Physics, University of Connecticut, Storrs, Connecticut 06268

Received August 16, 1983

We present a pedogogical review of Majorana masses and Majorana's theory of two-component massive fermions. We discuss the difference between Majorana and Dirac masses and show that Majorana masses are fermion-number violating. We discuss the connection between Majorana and Weyl spinors and show that the massive Majorana and Weyl field theories are equivalent. We study the second quantization of the massive Weyl theory in detail.

1. INTRODUCTION

Majorana's theory of the neutrino (Majorana, 1937) has proven to be both a basic development in the theory of fermions and an idea whose practicality has become apparent only many years after the original work itself, and then, moreover, in several different guises. Interest in the Majorana theory was revived some 20 years after its development following the discovery of parity violation and an associated two-component neutrino structure (see, e.g., Kabir, 1963); and interest was generated again some 20 more years later with the development of unified gauge theories (see, e.g., Frampton, 1980). Because of the fundamental significance of Majorana spinors and because of the current interest in Majorana masses we present here a pedagogical review of the subject.

In setting up the theory we study first the kinematics of c-number Majorana spinors. In Section 2 we discuss the Lorentz transformation properties of Majorana spinors and show that while they form an irreducible representation of the real Lorentz group, they decompose into a complex conjugate pair of Weyl spinors under the complex Lorentz group. Consequently the physical content of Majorana and Weyl spinors is the

0020-7748/84/0700-0643$03.50/0

same, and so Majorana and Weyl spinors can be used interchangeably. In Section 3 we establish the Lorentz invariance of a two-component Majorana or Weyl mass and show that such masses (unlike Dirac masses) are fermion-number violating. Thus we see that while two-component masses are not in fact forbidden by Lorentz invariance, they require fermion-number violation. For this reason they had been ignored for many years, and have only just recently come back into prominence because of the possibility of lepton-number violation in grandunified gauge theories (Frampton, 1980).

After discussing the kinematics of c-number Majorana spinors we proceed in Sections 4, 5, and 6 to study Majorana q-number fields. In Section 4 we show the equivalence of the Majorana and Weyl field theories. In Section 5 we study the general structure of the mass matrix in field theories where both Weyl and Dirac masses are simultaneously present and show that in general they admit of a pure Majorana structure after diagonalization. In Section 6 we present the canonical quantization of the Majorana theory.

Analogously to a c-number Klein–Gordon theory with real wave functions, the wave equations of the c-number Majorana theory also do not admit of any solutions which are energy eigenstates. However, again like the Klein–Gordon case, a successful particle interpretation is possible if we reinterpret the theory as a field theory. Thus the Majorana theory only exists as a second-quantized theory, so that second quantization is necessary from the beginning. We recall, in contrast, that in Dirac theory second quantization was only introduced at the end because the c-number theory was found to possess extra negative energy solutions. Hence for Dirac theory positive energy solutions do exist in the c-number theory and the subsequent filling of the negative energy sea then converts the theory into a q-number theory, whereas in the Majorana theory no positive (or negative) energy solutions exist at all until the theory is second-quantized. Thus second quantization is playing a much more central role here than in the Dirac case. Having identified the need for second quantization we are then able to provide a second-quantized definition of antiparticle as the hole in the negative energy sea. This definition is distinct from the c-number definition, which is the state coupled with the opposite sign to some external field. The two definitions do coincide for Dirac's electron but not for the massive neutrino, since no c-number theory exists for the latter. Thus the hole state definition is the more general.

To complete this review we present, in Section 7, a short discussion of the analogies between the theory of Majorana masses and Majorana's infinite component wave equation theory, and also discuss briefly some very recent applications of Majorana masses in modern gauge theories.

2. KINEMATICS OF MAJORANA SPINORS

In this section we discuss the kinematic properties of Majorana spinors, treating them in the first instance as state vectors, i.e., as first quantized c-number spinors represented by column vectors in an appropriate Dirac matrix space. A Majorana spinor, ψ^M, is defined via the relation

$$\psi^M = (\psi^M)^C \tag{1}$$

where ψ^C denotes conjugate spinor. A Majorana spinor is thus self-conjugate and hence a candidate for describing neutral particles such as the neutrino.

In order to discuss the transformation properties of Majorana spinors and to see their connection with the more familiar Weyl and Dirac spinors we shall find it convenient to use the Marjorana and Weyl bases for the Dirac gamma matrices rather than the standard Dirac basis

$$\gamma_0^D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \gamma_k^D = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}, \qquad \gamma_5^D = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2}$$

[throughout this paper we use the metric $g_{\mu\nu} = (1, -1, -1, -1)$]. The Majorana basis, which was introduced by Majorana himself in his original paper (Majorana, 1937), is defined by

$$\gamma_\mu^M = \frac{1}{\sqrt{2}}(1 - \gamma_2^D)\gamma_\mu^D \frac{1}{\sqrt{2}}(1 + \gamma_2^D) \tag{3}$$

so that

$$\gamma_0^M = \beta^M = \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix}, \qquad \gamma_1^M = -i\begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}, \qquad \gamma_2^M = \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix}$$

$$\gamma_3^M = i\begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix}, \qquad \gamma_5^M = \begin{pmatrix} -\sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix}, \qquad C^M = \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix} \tag{4}$$

$$\alpha_1^M = \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix}, \qquad \alpha_2^M = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \alpha_3^M = \begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix}$$

Analogously, the Weyl basis for the gamma matrices is defined by

$$\gamma_\mu^W = \frac{1}{\sqrt{2}}(1 - \gamma_5^D\gamma_0^D)\gamma_\mu^D \frac{1}{\sqrt{2}}(1 + \gamma_5^D\gamma_0^D) \tag{5}$$

so that

$$\gamma_0^W = \beta^W = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \qquad \gamma_k^W = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}$$

$$\alpha_k^W = \begin{pmatrix} \sigma_k & 0 \\ 0 & -\sigma_k \end{pmatrix}, \qquad \gamma_5^W = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad C^W = \begin{pmatrix} -i\sigma_2 & 0 \\ 0 & i\sigma_2 \end{pmatrix} \tag{6}$$

The charge conjugation and Lorentz transformation properties of Majorana spinors are most readily seen in the Majorana basis for the gamma matrices. In this basis charge conjugation only involves complex conjugation for states (columns) [or Hermitian conjugation for fields (matrices)] according to

$$\psi^C = \psi^* \tag{7}$$

so that there is no mixing of spinor components. Thus in this basis a Majorana spinor will be a four-component spinor each component of which is real to give a total of four independent real degrees of freedom. In the Majorana basis the Lorentz generators $\Sigma_{\mu\nu}$ $(=\gamma_\mu\gamma_\nu/2)$ have the special property of being real, i.e.,

$$\Sigma_{01} = \frac{1}{2}\begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix}, \qquad \Sigma_{02} = \frac{1}{2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \Sigma_{03} = \frac{1}{2}\begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix}$$

$$\Sigma_{12} = \frac{1}{2}\begin{pmatrix} 0 & -\sigma_1 \\ \sigma_1 & 0 \end{pmatrix}, \qquad \Sigma_{23} = \frac{1}{2}\begin{pmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{pmatrix},$$

$$\Sigma_{31} = \frac{1}{2}\begin{pmatrix} -i\sigma_2 & 0 \\ 0 & -i\sigma_2 \end{pmatrix} \tag{8}$$

and thus furnish a real representation of the commutation algebra

$$[\Sigma_{\mu\nu}, \Sigma_{\rho\sigma}] = -(g_{\mu\rho}\Sigma_{\nu\sigma} - g_{\nu\rho}\Sigma_{\mu\sigma} + g_{\mu\sigma}\Sigma_{\rho\nu} - g_{\nu\sigma}\Sigma_{\rho\mu}) \tag{9}$$

Under a Lorentz transformation an arbitrary Lorentz group spinor transforms as

$$\psi \to e^{\omega_{\mu\nu}\Sigma_{\mu\nu}}\psi \tag{10}$$

where $\omega_{\mu\nu}$ is real. Thus the phase of ψ is not changed under Lorentz transformations in the Majorana basis. Hence a Lorentz-transformed

Majorana spinor remains a Majorana spinor (in any basis), with the self-conjugacy condition of equation (1) being Lorentz invariant.

The representation of equation (8) is reducible, and can be reduced in the standard manner via

$$\Sigma^{\pm}_{\mu\nu} = \tfrac{1}{2}\left(1 \pm \gamma_5^M\right)\Sigma_{\mu\nu} \tag{11}$$

into its $D(1/2,0)$ and $D(0,1/2)$ pieces. Noting that γ_5^M is complex we thus see that even though $\Sigma_{\mu\nu}$ is real, $\Sigma^{\pm}_{\mu\nu}$ are complex. Thus $\Sigma_{\mu\nu}$ can only be reduced by allowing complex numbers. Hence even though $\Sigma_{\mu\nu}$ is reducible under the complex $SO(3,1;C)$ Lorentz group, it is nonetheless irreducible under the real $SO(3,1;R)$ Lorentz group alone. Consequently we see that a Majorana spinor transforms as a four-dimensional irreducible representation of $SO(3,1;R)$. With regard to this real representation of the Lorentz group we recall in passing that the commutation relations of equation (9) also admit of another purely real representation, the familiar $D(1/2,1/2)$ representation which acts on the coordinates x_μ, viz.,

$$\Sigma_{01} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \Sigma_{02} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\Sigma_{03} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad \Sigma_{12} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\Sigma_{23} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \Sigma_{31} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \tag{12}$$

so that altogether $SO(3,1;R)$ admits of two inequivalent real irreducible four-dimensional representations. This is in contrast to $SO(4;R)$ which only admits of one real irreducible four-dimensional representation, the one which acts on the coordinates. Hence the existence of a real Majorana spinor representation is associated with the Minkowski nature of the metric.

Once we have defined Majorana spinors we can then use them as a basis for constructing Dirac spinors. Along with ψ^M we introduce a second independent (anti) Majorana spinor $\psi^A[=-(\psi^A)^C]$ to give a total of eight real degrees of freedom. We define

$$\psi^D = \psi^M + \psi^A$$
$$(\psi^D)^C = \psi^M - \psi^A \tag{13}$$

so that ψ^M and ψ^A are indeed Majorana spinors, i.e.,

$$\psi^M = \tfrac{1}{2}\left[\psi^D + (\psi^D)^C\right] = (\psi^M)^C$$
$$\psi^A = \tfrac{1}{2}\left[\psi^D - (\psi^D)^C\right] = -(\psi^A)^C \tag{14}$$

Since the phase of an arbitrary spinor is unchanged by the Lorentz transformations of equation (10) in the Majorana basis we see that the decomposition of equation (13) is Lorentz invariant with ψ^D transforming as a spinor under the Lorentz group. Thus we recognize its eight real degrees of freedom as a complex four-component Dirac spinor.

As well as its decomposition into two independent self-conjugate Majorana spinors given in equation (13), the four-component Dirac spinor also admits of another decomposition, this one with respect to two independent complex Weyl spinors. Specifically, the complex two-component right- and left-handed Weyl spinors, ψ_R^W and ψ_L^W, transform according to the $D(1/2,0)$ and $D(0,1/2)$ representations of the Lorentz group and reduce the $D(1/2,0)\oplus D(0,1/2)$ Dirac spinor representation as

$$\psi^D = \psi_R^W + \psi_L^W \tag{15}$$

where

$$\psi_R^W = \tfrac{1}{2}(1+\gamma_5)\psi^D$$
$$\psi_L^W = \tfrac{1}{2}(1-\gamma_5)\psi^D \tag{16}$$

The transformation properties of the Weyl spinors are most easily seen in the Weyl basis for the gamma matrices given in equations (6). In the Weyl basis $(1\pm\gamma_5)/2$ project out the two upper and two lower components of ψ^D, while the Lorentz rotations $-\alpha_i\alpha_j/2$ and boosts $\alpha_k/2$ are block diagonal. Hence we see directly that ψ_R^W and ψ_L^W indeed transform as separate irreducible $D(1/2,0)$ and $D(0,1/2)$ representations under the homogeneous Lorentz group. However, the Weyl spinors are connectable by a discrete parity transformation. Consequently, a neutrino which transforms as ψ_L^W only, say, would give rise to the familiar parity-violating two-component neutrino theory.

To see the connection between Majorana spinors and Weyl spinors it is again convenient to use the Weyl basis for the gamma matrices. In this basis charge conjugation is given by

$$\psi^C = C^W \gamma_0^W \psi^* = i\gamma_2^W \psi^* = \begin{pmatrix} 0 & i\sigma_2 \\ -i\sigma_2 & 0 \end{pmatrix} \psi^* \tag{17}$$

where C^W is the matrix which transposes the γ_μ, i.e.,

$$C^{-1}\gamma_\mu C = -\gamma_\mu^T \tag{18}$$

Thus an arbitrary Majorana spinor can be written in the following equivalent forms:

$$\begin{aligned}\psi^M = \frac{1}{\sqrt{2}}\begin{pmatrix} a \\ b \\ -b^* \\ a^* \end{pmatrix} &= \frac{1}{\sqrt{2}}\begin{pmatrix} a \\ b \\ 0 \\ 0 \end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix} a \\ b \\ 0 \\ 0 \end{pmatrix}^C = \frac{1}{\sqrt{2}}\left[\psi_R^W + (\psi_R^W)^C\right] \\ &= \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 0 \\ -b^* \\ a^* \end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 0 \\ -b^* \\ a^* \end{pmatrix}^C = \frac{1}{\sqrt{2}}\left[\psi_L^W + (\psi_L^W)^C\right]\end{aligned} \tag{19}$$

with its four real degrees of freedom being expressed as the two complex quantities a and b. Analogously an arbitrary ψ^A spinor can be written as

$$\psi^A = \frac{1}{\sqrt{2}}\begin{pmatrix} c \\ d \\ d^* \\ -c^* \end{pmatrix} \tag{20}$$

We shall denote by $\hat{\psi}^M$ and $\hat{\psi}^A$ the particular pair of Majorana spinors for which $c = a$ and $d = b$. With these spinors we may now construct arbitrary Weyl spinors, since

$$\frac{1}{\sqrt{2}}(\hat{\psi}^M + \hat{\psi}^A) = \begin{pmatrix} a \\ b \\ 0 \\ 0 \end{pmatrix} = \psi_R^W = (\psi_L^W)^C \tag{21}$$

$$\frac{1}{\sqrt{2}}(\hat{\psi}^M - \hat{\psi}^A) = \begin{pmatrix} 0 \\ 0 \\ -b^* \\ a^* \end{pmatrix} = \psi_L^W = (\psi_R^W)^C \tag{22}$$

Thus starting from general Majorana spinors ψ^M and ψ^A we can build any arbitrary Dirac or Weyl spinor. Hence the Majorana spinors are building blocks for all the other spin-1/2 spinors.

From equations (19) we note further that a Majorana spinor can be interpreted either as a linear superposition of ψ_R^W and its own conjugate, or equivalently as a linear superposition of ψ_L^W and its own conjugate. Under Lorentz rotations and boosts the complex components a and b of ψ_R^W form a doublet and transform as $(\sigma, i\sigma)$. Thus $-i\sigma_2(\psi_R^W)^*$ transforms as $(\sigma, -i\sigma)$, i.e., the $-b^*, a^*$ doublet of $(\psi_R^W)^C$ transforms the same way as ψ_L^W. Hence ψ^M can be reexpressed as two pieces which transform as $D(1/2,0)$ and $D(0,1/2)$, respectively, and which are conjugate to each other. It is important to note that since the decomposition of equation (19) consists of a $D(1/2,0)$ representation and its own conjugate [as opposed to some arbitrary other $D(0,1/2)$ representation], it contains the same number of degrees of freedom as the original ψ^M, and thus has not changed the physical content. Thus self-conjugate Majorana spinors can be reexpressed as a pair of complex conjugate Weyl spinors so that Majorana and Weyl spinors are interchangeable and are equivalent in the sense of equation (19). Since it is inconvenient to work with the decomposition of equation (19) we shall formulate the theory of Majorana masses entirely in terms of Weyl spinors in the following.

3. MAJORANA AND WEYL MASSES AND FERMION-NUMBER NONCONSERVATION

The current interest in Majorana and Weyl spinors stems from the fact that it is possible to construct Lorentz invariant mass terms from them. We shall formulate the discussion initially in terms of Majorana spinors. In the direct product of two ψ^M spinors we can form a Lorentz scalar since

$$D(1/2,0) \otimes D(1/2,0) = D(0,0) \oplus D(1,0) \tag{23}$$

The scalar $D(0,0)$ term takes the form

$$(\psi^M)^T C \psi^M = \overline{(\psi^M)^C}\, \psi^M = \overline{\psi^M}\, \psi^M \tag{24}$$

and is known as a Majorana mass. To check that this term is a Lorentz scalar we note that under an infinitesimal Lorentz transformation $\delta\psi = \varepsilon_{\mu\nu}\gamma_\mu\gamma_\nu\psi$ the change in $\psi^T C \psi$ is

$$\varepsilon_{\mu\nu}\psi^T\left(C\gamma_\mu\gamma_\nu + \gamma_\nu^T\gamma_\mu^T C\right)\psi \tag{25}$$

which vanishes according to equation (18). To simplify the structure of the Majorana mass term we express ψ^M in the Weyl basis for the gamma matrices as

$$\psi^M = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi \\ -i\sigma_2\phi^* \end{pmatrix} \tag{26}$$

where ϕ denotes the two-component spinor

$$\phi = \begin{pmatrix} a \\ b \end{pmatrix} \tag{27}$$

With this definition we find that

$$\begin{aligned} m\overline{\psi^M}\psi^M &= \frac{m}{2}\phi^\dagger i\sigma_2\phi^* - \frac{m}{2}\phi^T i\sigma_2\phi \\ &= \frac{m}{2}[a^*b^* - b^*a^* - ab + ba] \end{aligned} \tag{28}$$

so that a Majorana mass is only nonvanishing if a and b are antisymmetric under interchange. Thus in a field theory a and b would have to be anticommuting fermion fields, while in a purely c-number theory a and b would have to be members of a Grassmann algebra and satisfy

$$\begin{aligned} \{a, a\}_+ &= \{b, b\}_+ = \{a, b\}_+ \\ &= \{a, a^*\}_+ = \{b, b^*\}_+ = \{a, b^*\}_+ = 0 \end{aligned} \tag{29}$$

As well as the above Majorana mass we can also form a Weyl mass since the direct product of two ψ_R^W (or two ψ_L^W, of course) spinors also contains a Lorentz scalar. In terms of Weyl spinors the scalar $D(0,0)$ term takes the form

$$\overline{(\psi_R^W)^C}\psi_R^W = (\psi_R^W)^T C\psi_R^W \tag{30}$$

In the Weyl basis for the gamma matrices ψ_R^W is given by

$$\psi_R^W = \begin{pmatrix} a \\ b \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \phi \\ 0 \\ 0 \end{pmatrix} \tag{31}$$

Thus the Weyl mass takes the form

$$\frac{m}{2}\left[(\psi_R^W)^T C^W \psi_R^W + (\psi_R^W)^\dagger (C^W)^\dagger (\psi_R^W)^*\right] = \frac{m}{2}\left[\phi^\dagger i\sigma_2\phi^* - \phi^T i\sigma_2\phi\right] \tag{32}$$

to thus establish the complete equivalence of the Weyl and Majorana masses.

Given the above form for the Weyl mass exhibited in equation (30) we now note some of its general properties. Since C^W is diagonal in the $D(1/2,0)$, $D(0,1/2)$ basis it only couples ψ_R^T to ψ_R or ψ_L^T to ψ_L, so it decomposes $\psi^T C\psi$ according to helicity, as is to be expected since C^W commutes with γ_5^W. Additional insight into the structure of the Weyl mass is obtained by recalling some familiar kinematic properties of a pair of identical fermions. For a pair of fermions with zero total angular momentum the available (l,s) states are (0,0) and (1,1). The (1,1) combination corresponds to $\psi^T C\gamma_\mu \partial_\mu \psi$ and the (0,0) to $\psi^T C\psi$. Now two fermions with zero total linear three momentum travel back to back. Their total spin wave function in the antisymmetric $s=0$ state is one in which the spins are antiparallel with respect to a fixed axis. Hence each particle has its spin parallel to its own direction of motion, to confirm that $\psi^T C\psi$ is diagonal in the helicity basis. Finally, we note that since the (l,s) part of the wave function of the (0,0) pair is antisymmetric the internal wave function of the pair must be symmetric. This will become a nontrivial restriction on Majorana masses when the fermions are placed in representations of non-Abelian unifying gauge groups.

As we thus see, the restriction to a two-component theory does not require the absence of mass terms, despite the somewhat widespread belief to the contrary. However, we note that $\psi^T C\psi$ transforms like a difermion and hence changes fermion number by two units. Thus the additional requirement of fermion number conservation (or of whatever other specific additive quantum numbers ψ carries) is needed, and usually assumed, in order to prevent Weyl spinors from acquiring a mass (Pauli, 1957). For this reason $\psi^T C\psi$ has not usually been considered as a candidate mass term in the literature. However, with the recent development of unified gauge theories with possible lepton number violation, interest has been revived in such mass terms. Thus what was once thought of as a vice of two-component neutrino mass terms, namely lepton number violation, has recently reemerged as a virtue.

Having now obtained the mass terms of equations (24) and (30) we can construct Lorentz invariant Lagrangians for Majorana and Weyl spinors, viz.,

$$\mathscr{L}_M = \frac{i}{2}\,\overline{\psi^M}\gamma_\mu \overleftrightarrow{\partial}_\mu \psi^M - m\,\overline{\psi^M}\psi^M \tag{33}$$

$$\mathscr{L}_W = \frac{i}{2}\,\overline{\psi_R^W}\gamma_\mu \overleftrightarrow{\partial}_\mu \psi_R^W - \frac{m}{2}\left[(\psi_R^W)^T C\psi_R^W + (\psi_R^W)^\dagger C^\dagger (\psi_R^W)^*\right] \tag{34}$$

From equations (26) and (31) and the Grassmann properties of equations (29) we find that both of these Lagrangians take the form

$$\mathscr{L} = \frac{i}{2}\phi^\dagger\left(\overleftrightarrow{\partial}_0 + \sigma_k \overleftrightarrow{\partial}_k\right)\phi + \frac{m}{2}\phi^T i\sigma_2\phi - \frac{m}{2}\phi^\dagger i\sigma_2\phi^* \tag{35}$$

to establish the complete equivalence of the Majorana and Weyl c-number spinor theories (Serpe, 1952; McLennan, 1957; Case, 1957).

The Lagrangian of equation (35) is now conveniently written entirely in terms of the four independent real degrees of freedom contained in the complex two-component spinor ϕ. The Euler-Lagrange variation of these Grassmann variables is straightforward and yields the equations of motion

$$\begin{aligned} (i\partial_0 + i\sigma_k\partial_k)\phi &= mi\sigma_2\phi^* \\ \phi^\dagger\left(i\overleftarrow{\partial}_0 + i\sigma_k\overleftarrow{\partial}_k\right) &= mi\phi^T\sigma_2 \end{aligned} \tag{36}$$

From equations (36) we obtain

$$\left(\partial_0^2 - \partial_k^2 + m^2\right)\phi = 0 \tag{37}$$

so that each component of ϕ satisfies the Klein–Gordon equation for a particle of mass m.

It is useful to compare the Majorana and Weyl spinors with the standard Dirac spinor, ψ^D, whose Lagrangian takes the form

$$\mathscr{L}_D = \frac{i}{2}\,\overline{\psi^D}\,\gamma_\mu\overleftrightarrow{\partial}_\mu\psi^D - m\,\overline{\psi^D}\,\psi^D \tag{38}$$

According to equation (13) we may decompose an arbitrary Dirac spinor into a pair of Majorana spinors, viz. (in the Weyl basis),

$$\psi^D = \frac{1}{\sqrt{2}}\begin{pmatrix}\phi \\ -i\sigma_2\phi^*\end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix}\rho \\ i\sigma_2\rho^*\end{pmatrix} \tag{39}$$

Under this decomposition the Dirac Lagrangian takes the form

$$\begin{aligned} \mathscr{L}_D = {} & \frac{i}{2}\phi^\dagger\left(\overleftrightarrow{\partial}_0 + \sigma_k\overleftrightarrow{\partial}_k\right)\phi + \frac{m}{2}\phi^T i\sigma_2\phi - \frac{m}{2}\phi^\dagger i\sigma_2\phi^* \\ & + \frac{i}{2}\rho^\dagger\left(\overleftrightarrow{\partial}_0 + \sigma_k\overleftrightarrow{\partial}_k\right)\rho - \frac{m}{2}\rho^T i\sigma_2\rho + \frac{m}{2}\rho^\dagger i\sigma_2\rho^* \end{aligned} \tag{40}$$

to again confirm the Lorentz irreducibility of equation (13). From equation

(40) we see that each Majorana spinor in equation (39) has its own associated Majorana Lagrangian, with the two Majorana masses reducing one Dirac mass. However, the reverse is not always true since we can only combine two Majorana masses into one Dirac mass as above provided the two masses are degenerate. Hence Majorana or Weyl masses are in principle different from Dirac masses.

We also note some additional kinematic distinctions between Weyl and Dirac masses. Under chiral transformations $\psi_R^T C\psi_R$ is purely right-handed and $\psi_L^T C\psi_L$ is purely left-handed, whereas the Dirac mass $\bar{\psi}\psi = \bar{\psi}_R\psi_L + \bar{\psi}_L\psi_R$ involves both helicities and transforms nontrivially with respect to both left- and right-handed chiral transformations. Specifically, if for example ψ belongs to the fundamental of an $SU(n)_L \times SU(n)_R$ group then the fermions transform as

$$\begin{aligned} \psi_L &\sim (n,1) \\ \psi_R &\sim (1,n) \\ \psi_D &\sim (n,1)\oplus(1,n) \end{aligned} \tag{41}$$

while the mass terms transform as (recall that Weyl masses are symmetric in the internal indices)

$$\begin{aligned} \psi_R^T C\psi_R &\sim (1, n(n+1)/2) \\ \psi_L^T C\psi_L &\sim (n(n+1)/2, 1) \\ \overline{\psi^D}\,\psi^D &\sim (n, n^*)\oplus(n^*, n) \end{aligned} \tag{42}$$

and thus the mass terms have totally different chiral properties. Additionally, each Weyl mass is parity violating while the Dirac mass is parity conserving. Finally, we note that each Weyl mass carries two units of fermion number while the Dirac mass has zero fermion number. Thus the various mass terms are qualitatively very different.

To complete our discussion of the c-number Majorana theory we also examine the solutions to the equations of motion of equation (36). In the rest frame we find that equation (36) possesses two solutions, viz.,

$$\phi_1 = \begin{pmatrix} \theta e^{-imt} \\ \theta^* e^{+imt} \end{pmatrix}, \qquad \phi_2 = \begin{pmatrix} -\theta^* e^{+imt} \\ \theta e^{-imt} \end{pmatrix} \tag{43}$$

where we have introduced convenient Grassmann variables θ and θ^* which satisfy

$$\{\theta, \theta\}_+ = \{\theta^*, \theta^*\}_+ = \{\theta, \theta^*\}_+ = 0 \tag{44}$$

Neither solution is an eigenstate of $i\partial_0$, i.e., neither solution satisfies

$$i\frac{\partial}{\partial t}\phi_i = \pm m\phi_i \tag{45}$$

The c-number theory therefore does not possess any traveling wave solutions which are eigenstates of $i\partial_0$. Since we would like to identify traveling waves as particles we see that this is impossible since we cannot form states of a definite energy. Hence, unlike the Dirac case, the massive Weyl theory does not exist as a one-particle theory. As we recall, the real Klein–Gordon wave equation does not possess any solutions of the form $\exp[i\bar{k}\cdot\bar{x} - i\omega t]$ when the wave function is real, and thus also has no one-particle limit. However, a successful particle interpretation of the Klein–Gordon theory was given by Pauli and Weisskopf following second quantization, an analysis which influenced Majorana in his setting up of an analogous real spinor theory. And indeed, as we shall see below in Section 6, the massive Majorana theory will also admit of sensible particle states following its reinterpretation as a second-quantized field theory. Thus the c-number Majorana theory has to be second-quantized.

Finally, we note a peculiarity of the Weyl mass. As is well known, the massless $m = 0$ Weyl equation does in fact admit of one-particle plane wave solutions, one of positive energy and the other of negative energy. When we add a mass term to the Weyl equation we find that we are unable to combine these two solutions into the two components of a spin-1/2 positive energy massive fermion, as is exhibited in the solutions of equation (43). Thus it is the mass term which forces the Weyl theory to become analogous to the real Klein–Gordon case, and hence it is the mass term which forces second quantization upon us.

4. EQUIVALENCE OF THE MAJORANA AND WEYL FIELD THEORIES

In setting up a q-number field theory we reinterpret the c-number fields as operators in an infinite dimensional Hilbert space and replace the Grassmann anticommutators of equations (29) by the standard equal time anticommutation relations for fermion fields. Apart from these changes demonstration of the equivalence of the Majorana and Weyl field theories essentially follows the previous c-number discussion. A Lagrangian density for a second-quantized Weyl field is given as in equation (34), viz.,

$$\mathscr{L} = \frac{i}{2}\overline{\psi_R^W}\gamma_\mu\overleftrightarrow{\partial}_\mu\psi_R^W - \frac{m}{2}\left[\psi_R^W C\psi_R^W + (\psi_R^W)^\dagger C^\dagger(\psi_R^W)^\dagger\right] \tag{46}$$

which takes the form

$$\mathscr{L} = \frac{i}{2}\phi^\dagger(\overleftrightarrow{\partial}_0 + \sigma_k\overleftrightarrow{\partial}_k)\phi + \frac{m}{2}\phi i\sigma_2\phi - \frac{m}{2}\phi^\dagger i\sigma_2\phi^\dagger \tag{47}$$

(in the Weyl basis for the gamma matrices) when we write

$$\psi_R^W = \begin{pmatrix}\phi\\0\\0\end{pmatrix} \tag{48}$$

We now introduce a new field[1]

$$\chi = \frac{1}{\sqrt{2}}\left[\psi_R^W + (\psi_R^W)^C\right] \tag{49}$$

which satisfies

$$\chi = \chi^C \tag{50}$$

Using the fermion field anticommutation relations we find that we can reexpress equation (46) as

$$\mathscr{L} = \frac{i}{2}\bar{\chi}\gamma_\mu\overleftrightarrow{\partial}_\mu\chi - m\bar{\chi}\chi \tag{51}$$

which we recognize as the Lagrangian for a self-conjugate Majorana field. With use of the transformation of equation (49) we thus show the equivalence of the Majorana and Weyl field theories.

The transformation of equation (49) is somewhat reminiscent of the Pauli–Gursey transformations (Pauli, 1957; Gursey, 1958)

$$\psi \to a\psi + b\gamma_5\psi^C \tag{52}$$

where

$$|a|^2 + |b|^2 = 1 \tag{53}$$

[1]The author is indebted to Professor N. G. Deshpande for explicitly identifying this transformation for him.

maintains unitarity. The Pauli–Gursey transformations leave invariant the antisymmetrized kinetic energy of a free massless fermion. However, equation (52) implies the Lorentz invariant transformations

$$\psi_R \to a\psi_R + b(\psi_L)^C \tag{54}$$

where both sides transform as the $D(1/2,0)$ representation, and

$$\psi_L \to a\psi_L - b(\psi_R)^C \tag{55}$$

where both sides transform as the $D(0,1/2)$ representation, and thus differs from equation (49). The difference lies in the fact that equation (54) mixes together two totally unrelated $D(1/2,0)$ representations, i.e., it mixes different degrees of freedom [and analogously for equation (55)], while equation (49) merely rewrites a given number of degrees of freedom in a self-conjugate form. Equation (49) is thus not a transformation but only a decomposition of a real field into a complex conjugate pair, as we discussed in Section 2.

Having now shown the equivalence of the Majorana and Weyl field theories we note the advantages of the Weyl formulation. The Lagrangian of the Majorana fields of equation (51) involves a constraint $\chi = \chi^C$ while the Weyl Lagrangian of equations (46) and (47) does not. Thus the Weyl spinors provide us with a formulation of the theory in terms of unconstrained degrees of freedom which is much more convenient for studying Euler–Lagrange variation, etc. Second, the Weyl degrees of freedom are complex and are hence more convenient for implementing complex phase transformations. Since we are interested in embedding the theory in a non-Abelian gauge theory, all of the associated symmetry transformations can most easily be discussed with Weyl spinors. Finally, in such theories most other fields (particularly those of electrically charged particles) will be described by Weyl spinors and hence their coupling to Majorana fields will be most easily implemented in the Weyl formulation of Majorana fields. Thus in the following we shall study the second-quantization and particle content of the theory directly from equations (46) and (47) rather than use Majorana fields explicitly.

5. THE MOST GENERAL MAJORANA, WEYL, AND DIRAC MASS MATRIX

In this section we discuss the structure of the general mass matrix when all types of mass are present simultaneously. For a single species of a neutral particle (which we conveniently describe by two Weyl spinors) the most general bilinear Lagrangian is

$$\begin{aligned}\mathscr{L} = \frac{i}{2}\bar{\nu}_L\gamma_\mu\overleftrightarrow{\partial}_\mu\nu_L + \frac{i}{2}\bar{\nu}_R\gamma_\mu\overleftrightarrow{\partial}_\mu\nu_R - \chi(\bar{\nu}_L\nu_R + \bar{\nu}_R\nu_L)\\ - \Delta_L(\nu_L C\nu_L + \nu_L^\dagger C^\dagger\nu_L^\dagger)\\ - \Delta_R(\nu_R C\nu_R + \nu_R^\dagger C^\dagger\nu_R^\dagger)\end{aligned} \tag{56}$$

where we take the Dirac mass parameter χ and the left- and right-handed Weyl mass parameters Δ_L and Δ_R to be real. In terms of the left-handed Weyl spinor

$$N = (\nu_R)^C \tag{57}$$

we can reexpress the Lagrangian as

$$\begin{aligned}\mathscr{L} = \frac{i}{2}\bar{\nu}_L\gamma_\mu\overleftrightarrow{\partial}_\mu\nu_L + \frac{i}{2}\bar{N}\gamma_\mu\overleftrightarrow{\partial}_\mu N\\ - \tfrac{1}{2}\chi(NC\nu_L + \nu_L CN + \nu_L^\dagger C^\dagger N^\dagger + N^\dagger C^\dagger\nu_L^\dagger)\\ - \Delta_L(\nu_L C\nu_L + \nu_L^\dagger C^\dagger\nu_L^\dagger)\\ - \Delta_R(NCN + N^\dagger C^\dagger N^\dagger)\end{aligned} \tag{58}$$

This gives a mass matrix in the (ν_L, N) basis

$$M = \begin{pmatrix}\Delta_L & \chi/2\\ \chi/2 & \Delta_R\end{pmatrix} \tag{59}$$

which is diagonalized by

$$\begin{aligned}P = \cos\alpha\nu_L + \sin\alpha N\\ Q = -\sin\alpha\nu_L + \cos\alpha N\end{aligned} \tag{60}$$

where

$$\tan 2\alpha = \frac{\chi}{(\Delta_L - \Delta_R)} \tag{61}$$

In terms of P and Q the full Lagrangian takes the form

$$\mathscr{L} = \frac{i}{2}\overline{P}\gamma_\mu \overleftrightarrow{\partial}_\mu P - m_P(PCP + P^\dagger C^\dagger P^\dagger)$$
$$+ \frac{i}{2}\overline{Q}\gamma_\mu \overleftrightarrow{\partial}_\mu Q - m_Q(QCQ + Q^\dagger C^\dagger Q^\dagger) \tag{62}$$

where

$$m_P = \tfrac{1}{2}(\Delta_L + \Delta_R) - \tfrac{1}{2}\left\{(\Delta_L - \Delta_R)^2 + \chi^2\right\}^{1/2}$$
$$m_Q = \tfrac{1}{2}(\Delta_L + \Delta_R) + \tfrac{1}{2}\left\{(\Delta_L - \Delta_R)^2 + \chi^2\right\}^{1/2} \tag{63}$$

The transformations of equation (60) are particular forms of the Pauli–Gursey transformations of equation (52). Specifically if

$$\psi \to \cos\alpha\psi - \sin\alpha\gamma_5\psi^C \tag{64}$$

then

$$\psi_L \to \cos\alpha\psi_L + \sin\alpha(\psi_R)^C$$
$$\psi_R \to \cos\alpha\psi_R - \sin\alpha(\psi_L)^C \tag{65}$$
$$(\psi_R)^C \to -\sin\alpha\psi_L + \cos\alpha(\psi_R)^C$$

which we recognize as equations (60). Thus when both Dirac and Weyl masses are present we can diagonalize the complete quadratic part of the Lagrangian by a Pauli–Gursey transformation. The resulting Lagrangian of equation (62) contains two uncoupled massive Weyl fields and is particle number violating. Unlike the case in equation (40) the two Weyl fields are not degenerate and hence cannot be combined into a Dirac field. Since a Dirac field can always be brought to a basis in which it is particle number conserving [as in equation (38)] we see the theory will possess an observable particle number violation unless either

$$(\Delta_L - \Delta_R)^2 + \chi^2 = 0 \tag{66}$$

or

$$\Delta_L = \Delta_R = 0 \tag{67}$$

Thus when both Weyl and Dirac masses are present the physical content of the theory is that of the pure particle number violating Weyl (or Majorana) theory of equation (62).

The extension of the above analysis to many neutral particles is straightforward. For N species we have

$$\begin{aligned}\mathscr{L} = &\sum_i \left(\frac{i}{2} \overline{\nu_L^i} \gamma_\mu \overleftrightarrow{\partial}_\mu \nu_L^i + \frac{i}{2} \overline{N}_i \gamma_\mu \overleftrightarrow{\partial}_\mu N_i \right) \\ &- \frac{1}{2} \sum_{ij} \chi^{ij} \left(\nu_L^i C N_j + N_j C \nu_L^i + N_j^\dagger C^\dagger \nu_L^{i\dagger} + \nu_L^{i\dagger} C^\dagger N_j^\dagger \right) \\ &- \sum_{ij} \Delta_L^{ij} \left(\nu_L^i C \nu_L^j + \nu_L^{j\dagger} C^\dagger \nu_L^{i\dagger} \right) \\ &- \sum_{ij} \Delta_R^{ij} \left(N_i C N_j + N_j^\dagger C^\dagger N_i^\dagger \right) \end{aligned} \tag{68}$$

where only the symmetric parts of Δ_L^{ij} and Δ_R^{ij} contribute. The generalized $2N$-dimensional Pauli–Gursey transformation

$$\begin{pmatrix} \nu_L^i \\ N_i \end{pmatrix} \to \begin{pmatrix} A_{ij} & B_{ij}^* \\ -B_{ij} & A_{ij}^* \end{pmatrix} \begin{pmatrix} \nu_L^j \\ N_j \end{pmatrix} = \begin{pmatrix} P_i \\ Q_i \end{pmatrix} \tag{69}$$

leaves the kinetic energy invariant provided it is unitary, i.e., provided the N-dimensional matrices A and B satisfy

$$A^\dagger A + B^\dagger B = I, \qquad B^T A = A^T B \tag{70}$$

Under this transformation the complete Lagrangian may be brought to the form

$$\begin{aligned}\mathscr{L} = &\sum_i \left\{ \frac{i}{2} \overline{P_i} \gamma_\mu \overleftrightarrow{\partial}_\mu P_i - m(P_i)\left(P_i C P_i + P_i^\dagger C^\dagger P_i^\dagger \right) \right\} \\ &+ \sum_i \left\{ \frac{i}{2} \overline{Q_i} \gamma_\mu \overleftrightarrow{\partial}_\mu Q_i - m(Q_i)\left(Q_i C Q_i + Q_i^\dagger C^\dagger Q_i^\dagger \right) \right\} \end{aligned} \tag{71}$$

where $m(P_i)$ and $m(Q_i)$ are the eigenvalues of the $2N$-dimensional Hermitian mass matrix

$$M = \begin{pmatrix} \Delta_L^{ij} & \chi^{ij}/2 \\ \chi^{ij}/2 & \Delta_R^{ij} \end{pmatrix} \tag{72}$$

Thus the complete quadratic piece of the Lagrangian is diagonalized by $2N$ Weyl fields whose masses in general are all different. Thus again the theory will be particle number violating. Finally, the transformations of equation (69) also have to be made on the interaction Lagrangian as well. This will then cause it to become particle number violating in general and will lead to transitions between the mass eigenstates of equation (71).

6. THE MASSIVE WEYL THEORY AS A FIELD THEORY

For the Lagrangian of the two-component Weyl field ϕ_i $(i=1,2)$ we take

$$\mathscr{L} = i\phi_i^\dagger \sigma_{ij}^\mu \partial_\mu \phi_j + i\frac{m}{2}\phi_i \sigma_{ij}^2 \phi_j - i\frac{m}{2}\phi_i^\dagger \sigma_{ij}^2 \phi_j^\dagger \tag{73}$$

where $\sigma_\mu = (1, \sigma_k)$. This Lagrangian generates the same action as the Lagrangian of equation (47). Euler–Lagrange variation of equation (73) then leads to the equations of motion

$$\begin{aligned} i\sigma_{ij}^\mu \partial_\mu \phi_j &= i\partial_0 \phi_i + i\sigma_{ij}^k \partial_k \phi_j = im\sigma_{ij}^2 \phi_j^\dagger \\ i\partial_\mu \phi_j^\dagger \sigma_{ji}^\mu &= im\phi_j \sigma_{ji}^2 \end{aligned} \tag{74}$$

provided the variation $\delta\phi$ anticommutes with ϕ. Because of these equations of motion the unequal time anticommutators must take the form (Case, 1957)

$$\begin{aligned} \{\phi_i(x,t), \phi_j^\dagger(x',t')\} &= -\left(\frac{\partial}{\partial t} - \sigma_k \frac{\partial}{\partial x_k}\right)_{ij} D(x-x', t-t') \\ \{\phi_i(x,t), \phi_j(x',t')\} &= -m\sigma_{ij}^2 D(x-x', t-t') \end{aligned} \tag{75}$$

where

$$D(x,t) = -\frac{1}{(2\pi)^3}\int \frac{d^3k}{k_0} e^{i\bar{k}.\bar{x}} \sin k_0 t \tag{76}$$

and

$$k_0 = +\left(|\bar{k}|^2 + m^2\right)^{1/2} \tag{77}$$

These relations in turn give the standard equal time anticommutators

$$\{\phi_i(x,t),\phi_j^\dagger(x',t)\} = \delta_{ij}\delta^3(x-x')$$
$$\{\phi_i(x,t),\phi_j(x',t)\} = 0 \qquad (78)$$

An energy-momentum tensor is defined as

$$T_{\mu\nu} = \frac{\partial\mathscr{L}}{\partial\partial_\mu\phi}\frac{\partial\phi}{\partial x_\nu} - g_{\mu\nu}\mathscr{L} \qquad (79)$$

and takes the form

$$T_{\mu\nu} = i\phi^\dagger\sigma_\mu\partial_\nu\phi - i\frac{m}{2}g_{\mu\nu}(\phi\sigma_2\phi + \phi^\dagger\sigma_2\phi^\dagger) \qquad (80)$$

for fields which satisfy the equations of motion. [In such solutions we note, in passing, that, unlike the case for the familiar Dirac equation, the Lagrangian of equation (73) does not vanish.] Using the field equations and the anticommutation relations we then find that $T_{\mu\nu}$ is conserved,

$$\partial_\mu T_{\mu\nu} = 0 \qquad (81)$$

so that

$$P_\mu = \int d^3x\, T_{0\mu} \qquad (82)$$

are the translation generators. Thus the Hamiltonian takes the form

$$H = \frac{1}{2}\int d^3x\{i\phi^\dagger\overleftrightarrow{\partial}_0\phi\} \qquad (83)$$

and obeys

$$i[H,\phi] = \frac{\partial\phi}{\partial t} \qquad (84)$$

Having now obtained the field equations, to complete the analysis we must also construct the states of the theory so as to establish the particle content of the field operator $\phi(x,t)$. To make a normal mode expansion we follow Case (Case, 1957). We introduce a helicity basis for linear momentum $\bar{k}$ according to ($k = |\bar{k}|$)

$$\bar{\sigma}.\bar{k}\alpha(\bar{k}) = k\alpha(\bar{k})$$
$$\bar{\sigma}.\bar{k}\beta(\bar{k}) = -k\beta(\bar{k}) \qquad (85)$$

We can make an arbitrary momentum space expansion for $\phi(x,t)$ in terms of these helicity states and in terms of the positive and negative frequency solutions to the second derivative Klein–Gordon equation that $\phi(x,t)$ also satisfies, viz.,

$$\phi(x,t)=\frac{1}{V^{1/2}}\sum_{\bar{k}}\left(\frac{k_0+k}{2k_0}\right)^{1/2}$$
$$\times\left\{e^{-ik.x}\left[b_+(\bar{k})\alpha(\bar{k})-\frac{m}{(k_0+k)}b_-(\bar{k})\beta(\bar{k})\right]\right.$$
$$\left.+e^{+ik.x}\left[d_-^\dagger(\bar{k})\alpha(\bar{k})+\frac{m}{(k_0+k)}d_+^\dagger(\bar{k})\beta(\bar{k})\right]\right\} \quad (86)$$

where k_0 is defined in equation (77) and V is a volume factor. The requirement that $\phi(x,t)$ also satisfy the first derivative equations of motion, equations (74), then reduces the theory to two complex degrees of freedom by requiring

$$d_\pm(\bar{k})=b_\pm(\bar{k}) \quad (87)$$

Thus the most general solution to the equations of motion takes the form (Case, 1957)

$$\phi(x,t)=\frac{1}{V^{1/2}}\sum_{\bar{k}}\left(\frac{k_0+k}{2k_0}\right)^{1/2}$$
$$\times\left\{e^{-ik.x}\left[b_+(\bar{k})\alpha(\bar{k})-\frac{m}{(k_0+k)}b_-(\bar{k})\beta(\bar{k})\right]\right.$$
$$\left.+e^{+ik.x}\left[b_-^\dagger(\bar{k})\alpha(\bar{k})+\frac{m}{(k_0+k)}b_+^\dagger(\bar{k})\beta(\bar{k})\right]\right\} \quad (88)$$

The second quantization of $\phi(x,t)$ according to equation (78) yields

$$\{b_\pm(\bar{k}),b_\pm(\bar{k}')\}=\{b_\pm(\bar{k}),b_\mp(\bar{k}')\}$$
$$=\{b_\pm(\bar{k}),b_\mp^\dagger(\bar{k}')\}=0$$
$$\{b_\pm(\bar{k}),b_\pm^\dagger(\bar{k}')\}=\delta(\bar{k},\bar{k}') \quad (89)$$

so that $b_\pm^\dagger(\bar{k})$ have the conventional interpretation of creating particles out of the vacuum, $|0\rangle$.

In terms of the creation and annihilation operators the Hamiltonian of equation (83) takes the form

$$H = \sum_{\bar{k}} k_0 \left[b_+^\dagger(\bar{k}) b_+(\bar{k}) - b_-(\bar{k}) b_-^\dagger(\bar{k}) \right]$$
$$= \sum_{\bar{k}} k_0 \left[b_+^\dagger(\bar{k}) b_+(\bar{k}) - d_-(\bar{k}) d_-^\dagger(\bar{k}) \right] \tag{90}$$

which we can normal order as

$$H = \sum_{\bar{k}} k_0 \left[b_+^\dagger(\bar{k}) b_+(\bar{k}) + b_-^\dagger(\bar{k}) b_-(\bar{k}) - 1 \right] \tag{91}$$

Thus the two states $b_\pm^\dagger(\bar{k})|0\rangle$ are two positive energy eigenstates of the Hamiltonian of equation (91) and thus give a sensible particle interpretation to the field $\phi(x, t)$.[2]

It is important to notice that the kinematic coefficients of $b_\pm(\bar{k})$, $b_\pm^\dagger(\bar{k})$ in equation (88) are not themselves solutions to the c-number equations of motion, in contrast to the situation met in the standard Dirac and Klein–Gordon theories. Further, in the q-number theory, it is only the interplay of the positive and negative frequency parts of $\phi(x, t)$, defined via

$$\phi(x, t) = \phi_{(+)}(x, t) + \phi_{(-)}(x, t) \tag{92}$$

which enables $\phi(x, t)$ to satisfy the q-number equations of motion. The positive and negative frequency parts of the field operator satisfy

$$i\sigma_\mu \partial_\mu \phi_{(\pm)} = mi\sigma_2 \left[\phi_{(\mp)} \right]^\dagger \neq mi\sigma_2 \phi_{(\pm)}^\dagger \tag{93}$$

and thus, again unlike the standard Dirac and Klein–Gordon theories, are not themselves separate solutions to the field equations, but instead are connected with each other in equation (93) in a way which enables $\phi(x, t)$ to satisfy equation (74). As we recall, $\phi(x, t)$ is not itself a self-conjugate field. However, because of the identification of $b_\pm(\bar{k})$ with $d_\pm(\bar{k})$ in equation (87) we note that $\phi(x, t)$ and $\phi^\dagger(x, t)$ both create the same particle states out

[2] In passing we note that the Hamiltonian of equation (91) is invariant under parity transformations which mix the two helicity states. Because of equation (87) this is equivalent to a CP transformation on the fields, with the Lagrangian of equation (73) being CP invariant. Thus even though the Lagrangian of the free massive Weyl theory is not parity invariant in field space, there does exist a good parity transformation for the states. Hence any parity violation that might be observed in processes involving the scattering of these states would then have to be due to a parity noninvariance of the interactions and would not be related to the parity noninvariance of the free Lagrangian.

of the vacuum, so as far as its particle content is concerned $\phi(x,t)$ is acting like a self-conjugate field.

If we take the matrix element of equation (84) between the vacuum and a one-particle state at rest

$$|1\rangle = b_+^\dagger(\bar{0})|0\rangle \tag{94}$$

we obtain

$$\langle 0|\phi|1\rangle\langle 1|H|1\rangle - \langle 0|H|2\rangle\langle 2|\phi|1\rangle = i\frac{\partial}{\partial t}\langle 0|\phi|1\rangle \tag{95}$$

since $\phi(x,t)$ can only change particle number by one unit. Since the Hamiltonian of equation (91) is diagonal in this same particle number basis the matrix element $\langle 0|H|2\rangle$ is zero. [The nontrivial content of this remark, which we will explore below, is to note that the Lagrangian of equation (73) does change another particle number by two units, namely, the usual fermion number.] Consequently we conclude that the c-number wave function $\langle 0|\phi|1\rangle$ is an eigenstate of $i\partial_0$ with eigenvalue $\langle 1|H|1\rangle$, i.e.,

$$\begin{aligned} i\frac{\partial}{\partial t}\langle 0|\phi|1\rangle &= \langle 1|H|1\rangle\langle 0|\phi|1\rangle \\ &= m\langle 0|\phi|1\rangle \\ &\equiv me^{-imt}\alpha(\bar{0}) \end{aligned} \tag{96}$$

Thus the diagonalization of the Hamiltonian in an appropriate second-quantized basis enables us to construct the desired first-quantized energy eigenstates. Further, if we take the matrix element of the field equation between the vacuum and the state $|1\rangle$ we obtain

$$\begin{aligned} i\frac{\partial}{\partial t}\langle 0|\phi|1\rangle &= im\sigma_2\langle 0|\phi^\dagger|1\rangle \\ &= im\sigma_2\langle 1|\phi|0\rangle^* \end{aligned} \tag{97}$$

Thus the matrix element $\langle 0|\phi|1\rangle$, while being an eigenstate of $i\partial_0$, is not a solution to the naive one-particle equation which would be suggested by using equation (36) blindly, viz.,

$$i\frac{\partial}{\partial t}\langle 0|\phi|1\rangle = im\sigma_2\langle 0|\phi|1\rangle^* \tag{98}$$

precisely because ϕ is a matrix. Thus, unlike the situation met in standard field theories, the matrix element $\langle 0|\phi|1\rangle$ is simply not a solution to the c-number wave equation, and it is that fact which enables us to avoid the difficulties associated with the absence of plane wave solutions to the c-number theory discussed in Section 3.

To obtain the above particle interpretation of the second-quantized massive Weyl theory it was necessary for us to be able to diagonalize the Hamiltonian of equation (91) in a particle number conserving basis. Specifically we find that the operator

$$N=\sum_{\bar{k}}\left[b_{+}^{\dagger}(\bar{k})b_{+}(\bar{k})+b_{-}^{\dagger}(\bar{k})b_{-}(\bar{k})\right] \tag{99}$$

obeys

$$i[H,N]=\frac{\partial N}{\partial t}=0 \tag{100}$$

Thus the massive Weyl theory possesses a good conserved number operator N. Since the Lagrangian of equation (73) is not invariant under the familiar

$$\phi\rightarrow e^{i\alpha}\phi \tag{101}$$

fermion number phase transformations, N is not the usual fermion number. Thus it is of interest to explore the nature of N. To this end we express N in coordinate space. Inverting equation (88) and its Hermitian conjugate yields

$$N=\int d^3x\,d^3x'\,D_{(+)}(x'-x,0)\phi^{\dagger}(x',t)i\frac{\overleftrightarrow{\partial}}{\partial t}\phi(x,t) \tag{102}$$

Here we have introduced the familiar nonlocal propagator for positive frequency modes

$$D_{(+)}(x,t)=\frac{1}{(2\pi)^3}\int\frac{d^3k}{2k_0}e^{-ik_0t+i\bar{k}.\bar{x}} \tag{103}$$

where k_0, as previously, is given by equation (77). Thus the number operator N is a nonlocal operator and cannot therefore be associated with any local Noether variation of the Lagrangian. The operator N is a pure second-quantized object which is not obtainable by quantizing any conserved quantity associated with a local invariance of the classical Lagrangian. While the conservation of N is thus an interesting feature of the free massive

Weyl theory, since N is nonlocal it is unlikely that it could be conserved by interactions. Nonetheless, even in the presence of interactions N is still useful as it is needed as a basis for classifying asymptotic scattering states.

Given our above remarks we now note some interesting differences between the massive Weyl theory and the standard Dirac theory. As we recall, in Dirac theory the second-quantized fields can be expanded in a complete set of wave functions which are solutions to the one-particle Dirac equation; so that the vacuum to one particle matrix elements of the field operators then satisfy the same equations of motion as the fields themselves, namely the Dirac equation. So for the Dirac equation a sensible first-quantized theory does exist, with second quantization only being introduced essentially as an afterthought because the need to fill the negative energy sea *a posteriori* converted the theory into a many-body theory. Further, in Dirac theory the relevant particle number is a quantum number which is associated with a Noether invariance of the classical equations of motion, to thus put the modes of the field into one-to-one correspondence with the solutions to the c-number Dirac equation. For the massive Weyl theory on the other hand there is no such one-to-one correspondence between the first- and second-quantized theories (the solutions to the first-quantized equations are not the coefficients in the normal mode expansion of the second-quantized field); and the appropriate particle number operator of the second-quantized Weyl field theory is not obtained from a classical Noether analysis. Consequently for the massive Weyl theory second quantization is necessary *a priori* in order to produce a sensible particle interpretation.

The massive Weyl theory is thus seen to be a theory which has no associated c-number limit, and to be one for which the field aspect is necessary in order to establish the particle content of the theory. Thus an S matrix for the scattering of the states created by $\phi(x, t)$ cannot be constructed without a knowledge of the underlying field theory. Ordinarily, the S-matrix and Lagrangian descriptions of the scattering of asymptotic states are equivalent. The S-matrix approach only requires a knowledge of the c-number equations for the states (the one-particle Dirac equation for instance) so as to describe the kinematic behavior of the asymptotic states under the Lorentz group in order to impose relativistic covariance; and then appeals to unitarity in order to account for processes which change the number of particles. The Lagrangian approach, alternatively, uses the dynamical behavior of the fields themselves in order to change the number of particles. Since the fields usually satisfy the same equations as the c-number states the two approaches are usually indistinguishable in their predictions with it being possible for instance to formulate a pure S-matrix theory of quantum electrodynamics. In the massive Weyl theory we now discover a rather different situation, i.e., in the absence of any associated conventional one-particle theory the Lagrangian approach becomes essential for describing the on-shell scattering of asymptotic states. This is then an example of a theory where a field description has more content than a pure particle description.

As a field theory the massive Weyl theory also differs from Dirac theory in one other interesting way, namely, in its normal ordering prescription. For a standard four-component Dirac spinor a conventional normal mode expansion is

$$\psi^D(x,t)=\frac{1}{V^{1/2}}\sum_{\bar{k},\pm}\left\{e^{-ik.x}u_\pm(\bar{k})\hat{b}_\pm(\bar{k})+e^{+ik.x}v_\pm(\bar{k})\hat{d}^\dagger_\pm(\bar{k})\right\} \tag{104}$$

where $u_\pm(\bar{k})$, $v_\pm(\bar{k})$ are the familiar c-number Dirac spinors. The second quantization of the Dirac theory leads to the Hamiltonian

$$H=\sum_{\bar{k},\pm}k_0\left[\hat{b}^\dagger_\pm(\bar{k})\hat{b}_\pm(\bar{k})-\hat{d}_\pm(\bar{k})\hat{d}^\dagger_\pm(\bar{k})\right] \tag{105}$$

which we rewrite as

$$H=\sum_{\bar{k},\pm}k_0\left[\hat{b}^\dagger_\pm(\bar{k})\hat{b}_\pm(\bar{k})+\hat{d}^\dagger_\pm(\bar{k})\hat{d}_\pm(\bar{k})-1\right] \tag{106}$$

The constant term is then removed by normal ordering with respect to the filled negative energy sea, so that the Dirac theory treats positive and negative frequencies very differently. On the other hand, in the massive Weyl theory we also have to normal order the Hamiltonian of equation (90), but now we do so by occupying the states created by $b^\dagger_-(\bar{k})$, as indicated in equation (91), i.e., we normal order effectively with respect to positive frequencies [or equivalently with respect to negative frequencies according to equation (87)] so that now the positive and negative frequencies are treated symmetrically. We recall that this of course was one of the reasons Majorana originally set up the theory in the first place.

The above remarks also help clarify the meaning of antiparticle. There are actually two definitions of antiparticle, first the state coupled to an external electromagnetic field with the opposite sign, and second the state associated with a negative frequency mode. The first definition is a c-number single-particle definition, while the second definition is a q-number many-body definition. For the Dirac theory the two definitions coincide precisely because the theory has a c-number limit, with the c-number wave functions describing the same particles as those associated with the q-number fields. However, for the massive Weyl theory no such c-number limit exists, and thus a c-number definition of antiparticle is not even available.

[Additionally there is no conserved electromagnetic current in the two-component theory to which electromagnetism could couple (Case, 1957).] In the massive Weyl theory only the second-quantized definition of antiparticle survives, with the particle being its own antiparticle according to equation (87). Thus the second-quantized hole state definition is the more general as it allows us to introduce a concept of antiparticle even when no coupling to an external electromagnetic field is possible.

Finally in this context we comment on how the TCP theorem is satisfied. Under the combined TCP operation a four-component Dirac spinor transforms as

$$\theta\psi^D(x,t)\theta^{-1} = i\gamma_5\psi^{D\dagger}(-x,-t) \tag{107}$$

with the creation and annihilation operators of equation (104) transforming as

$$\theta\hat{b}(\bar{k},\pm)\theta^{-1} = \hat{d}(\bar{k},\mp) \tag{108}$$

so that TCP takes electrons into positrons. In the massive Weyl theory equations (107) and (108) reduce to

$$\theta\phi(x,t)\theta^{-1} = i\phi^\dagger(-x,-t) \tag{109}$$

and

$$\theta b(\bar{k},\pm)\theta^{-1} = b(\bar{k},\mp) \tag{110}$$

Hence the state $b^\dagger_+(\bar{k})|0\rangle$ has the same particle content as the state $\theta b^\dagger_+(\bar{k})|0\rangle$, as the particle is its own antiparticle. Further, the Lagrangian of equation (73) is invariant under the TCP transformation of equation (109). As we recall, if a Hamiltonian is TCP invariant, then any given eigenstate and its TCP transform are degenerate in mass. Ordinarily these two states are distinct, with the TCP theorem then providing for the degeneracy of a particle with its antiparticle. In the present massive Weyl theory the two states are states of one and the same particle so that the particle gets to be degenerate with its antiparticle by being its own antiparticle. The TCP structure of antiparticles is thus completely in accord with our previous analysis.

For completeness we conclude this section by evaluating the Feynman propagator of the free massive Weyl theory. Using the second-quantized form of $\phi(x,t)$ given in equation (88) we obtain directly

$$-i\langle 0|T\big(\phi_j(x')\phi_i^\dagger(x)\big)|0\rangle = \int \frac{d^4k}{(2\pi)^4} e^{ik.(x-x')} \frac{\sigma^0_{ji}k_0 + \bar{\sigma}_{ji}.\bar{k}}{k^2 - m^2 + i\varepsilon} \tag{111}$$

We note that the numerator of the massive Weyl propagator differs from the standard $\gamma^{\mu}p_{\mu} + m$ form of the Dirac propagator by the absence of any mass factor m. Apart from the fact that at the poles k_0 is given by $\pm(\bar{k}^2 + m^2)^{1/2}$ in equation (111) the kinematic structure of the numerator of the massive Weyl propagator is otherwise the same as that of the massless Weyl propagator. This missing m factor instead appears in the numerator of a second propagator of the massive Weyl theory, viz.,

$$-i\langle 0|T(\phi_j(x')\phi_i(x))|0\rangle = \int \frac{d^4k}{(2\pi)^4} e^{ik.(x-x')} \frac{im\sigma_{ji}^2}{k^2 - m^2 + i\varepsilon} \tag{112}$$

This latter propagator is nonvanishing because the massive Weyl theory is not invariant under the phase transformations of equation (101), and would thus yield additional contractions in the Feynman rules for interacting massive Weyl fields.

7. FINAL REMARKS

In looking back over all of Majorana's work in fundamental theory it would appear that Majorana was somewhat unhappy with Dirac theory, and in particular with the existence of negative energy solutions. Specifically, in his other well-known paper (Majorana, 1932) (which was written prior to his work on self-conjugate fermion fields) he had also made an attempt to address the negative energy question, by constructing a c-number theory for the electron which only possessed positive energy solutions, the infinite component wave equation theory. Historically, this work appeared at about the same time as the experimental discovery of the positron, indicating that in fact the electron was classified according to the $D(1/2,0)\oplus D(0,1/2)$ representation rather than according to an infinite-dimensional representation of the Lorentz group, so Majorana's work was largely ignored. Nonetheless it constitutes an interesting theoretical alternative for classifying spin-1/2 particles, and we shall review it briefly now, because, as we shall see, it bears some similarity to Majorana's self-conjugate theory.

To begin, we recall that a Majorana self-conjugate spinor satisfies $\psi = \psi^C$. In the Dirac basis for the gamma matrices this leads to the same relation as the Weyl basis relation given in equation (19), viz.,

$$\psi = \frac{1}{\sqrt{2}} \begin{pmatrix} a \\ b \\ -b^{\dagger} \\ a^{\dagger} \end{pmatrix} \tag{113}$$

In terms of this spinor we construct six operators

$$M_{ij} = \frac{i}{2}\bar{\psi}\gamma_i^D\gamma_j^D\psi$$
$$M_{0i} = \frac{i}{2}\bar{\psi}\gamma_0^D\gamma_i^D\psi \tag{114}$$

Now, unlike the previous fermion case, we instead take a and b to be operators that satisfy Bose commutation relations. In such a case we find

$$M_{12} = \tfrac{1}{2}(a^\dagger a - b^\dagger b), \qquad M_{23} = \tfrac{1}{2}(a^\dagger b + b^\dagger a), \qquad M_{31} = \frac{i}{2}(b^\dagger a - a^\dagger b)$$
$$M_{01} = \frac{i}{4}(a^{\dagger 2} - b^{\dagger 2} - a^2 + b^2), \qquad M_{02} = \tfrac{1}{4}(a^{\dagger 2} + b^{\dagger 2} + a^2 + b^2),$$
$$M_{03} = \frac{i}{2}(ab - a^\dagger b^\dagger) \tag{115}$$

Using the commutation relations again we find that these six operators close on the Lorentz algebra for $M_{\mu\nu}$ given in equation (9). Since the operators are also Hermitian they thus provide us with an infinite-dimensional unitary representation of the Lorentz group. (Had we instead used Fermi anticommutation relations the angular momentum changing boosts would have cut off at a finite point because of the Pauli principle.)

As well as the above six bilinear combinations of creation and annihilation operators there are four other combinations. We define them as

$$\Gamma_0 = \bar{\psi}\gamma_0^D\psi = (a^\dagger a + b^\dagger b + 1)$$
$$\Gamma_1 = \bar{\psi}\gamma_1^D\psi = \tfrac{1}{2}(a^{\dagger 2} - b^{\dagger 2} + a^2 - b^2)$$
$$\Gamma_2 = \bar{\psi}\gamma_2^D\psi = -\frac{i}{2}(a^{\dagger 2} + b^{\dagger 2} - a^2 - b^2)$$
$$\Gamma_3 = \bar{\psi}\gamma_3^D\psi = -(a^\dagger b^\dagger + ab) \tag{116}$$

The significance of these four infinite-dimensional Γ_μ is that they transform as a four-vector under the generators of equation (115), viz.,

$$[M_{\mu\nu}, \Gamma_\sigma] = -i(g_{\mu\sigma}\Gamma_\nu - g_{\nu\sigma}\Gamma_\mu) \tag{117}$$

Consequently, if we introduce an infinite-dimensional c-number column vector, χ, then its c-number equation of motion

$$(i\Gamma_\mu \partial_\mu - m)\chi = 0 \tag{118}$$

is immediately covariant. Equation (118) is known as the Majorana infinite-component wave equation. It differs from the Dirac equation in that the quantities Γ_μ do not form a Dirac gamma matrix algebra, so that the components of χ do not satisfy the Klein–Gordon equation. From equation (116) we see that Γ_0 is diagonal with all of its diagonal elements being positive definite. Thus equation (118) only possesses positive energy solutions, this being Majorana's original objective. Further, since the boosts change angular momentum by two half units the complete multiplet of χ contains a tower of spins which take on all half-integer values. Thus the absence of one negative energy spin one-half particle requires instead a whole infinite tower of half-integer spin positive energy states.

The Majorana equation suffers from a well-known disease, namely, that it also possesses spacelike solutions. In our notation this may be seen directly. Specifically we note that Γ_3 is Hermitian. If we solve equation (118) in a Lorentz frame in which only p_3 is nonzero then after rediagonalization we will find that the solutions behave like $p_3^2 \sim +m^2$, and thus correspond to spacelike vectors. [The corresponding situation for the usual Dirac equation is that γ_3^D of equation (2) is anti-Hermitian so that then we would obtain $p_3^2 \sim -m^2$ to only allow timelike solutions.]

We thus see that there are many parallels between Majorana's two attempts to eliminate negative energies. Both the infinite component theory and the self-conjugate theory are candidate theories for the classification of a spin-1/2 particle which are alternatives to Dirac theory. Both of them are based on Majorana spinors, one with boson quantization and the other with fermion quantization. Both contain (and need) an infinite number of degrees of freedom, one through being an infinite component c-number theory, and the other through being a second-quantized q-number theory. Finally both of them contain spacelike propagation, one through having spacelike solutions to the equations of motion, and the other through possessing a quantum number [N of equation (102)] whose conservation would require information being communicated between spacelike separated points. Because of this last feature both attempts are perhaps a little troubling, but the full implications of the spacelike structures remain to still be explored.

Apart from attacking the basic theoretical question of the general nature of fermions, Majorana's self-conjugate theory has also very recently come back into prominence because of developments in modern gauge theories. There are applications in both pure electroweak unification models

(Mannheim, 1979; 1980) and also in grandunification models (Frampton, 1980). The central questions that are raised are whether the physical neutrinos are two-component or four-component, whether they are massive or massless, and if massive whether they acquire Dirac or Majorana masses or both, whether lepton number is a global or a local symmetry, and finally whether it is broken in the neutrino sector. There appear to be models to cover each eventuality, so we discuss here very briefly only a few ideas which have a more theoretical as opposed to phenomenological basis.

The most interesting question regarding neutrinos is whether they possess two or four components. While there is as yet no experimental indication that the neutrino is a four component spinor (recall that even the observation of a nonvanishing neutrino mass may merely be an indication that the neutrino is a massive two-component Weyl spinor), nonetheless the use of the right-handed neutrinos has some theoretical advantages. It was pointed out (Mannheim, 1979; 1980) that they could play a very useful role in $SU(2)_L \times SU(2)_R \times U(1)_V$ extensions of the Weinberg–Salam theory. In such theories the $U(1)_V$ generator is the operator $B_V - L_V$, baryon number minus lepton number, so lepton number plays an explicit role. Breaking the theory by giving the right-handed neutrinos a Majorana mass (i.e., breaking as $\nu_R^T C \nu_R$) exactly breaks the theory down to the (so far exact) Weinberg–Salam model. Thus the asymmetry implicit in starting with $SU(2)_L \times U(1)$ is completely removed in both the charged and neutral current sectors by spontaneously breaking the left–right symmetric theory according to only one specific term, a right-handed neutrino Majorana mass, which is both compact and elegant. The subsequent breaking of $SU(2)_L \times U(1)$ down to electromagnetism then produces the usual Weinberg mixing pattern for the neutral currents provided that the left-handed neutrinos do not also acquire a Majorana mass. To ensure this Mannheim's model retained a residual unbroken global current $L_V - N_R = L_{WS}$ ($= \bar{e}\gamma_\lambda e + \overline{\nu_L^e}\gamma_\lambda \nu_L^e$ for the first family) which we recognize as the conventionally defined lepton number of the usual Weinberg–Salam weak interactions. Further, the conservation of this current also prevents the neutrinos from acquiring a Dirac mass so that the left-handed neutrinos are completely massless. The lepton number L_V is thus only broken in the right-handed neutrino sector, and that is sufficient to produce all the standard Weinberg–Salam phenomenology while pushing the unwanted lepton number violations (which lead to processes such as neutrinoless beta decay) into the so far unobserved right-handed sector of the theory. Moreover in such a picture the right-handed neutrinos need only acquire typical regular light fermion masses to avoid any conflict with present experimental phenomenology. Thus we see the practicality of Majorana mass breaking in pure electroweak unification models.

Majorana masses are also prominent in grandunification models (Frampton, 1980). There, the usual baryon and lepton number operators (or linear combinations of them) are gauged with the Weinberg–Salam theory being embedded in some large grandunifying group. Baryon and lepton number are then typically broken at the superheavy 10^{15} GeV mass scale to leave the Weinberg–Salam model as an approximate residual light symmetry. One of the most interesting situations occurs in the widely discussed $SO(10)$ model. There the right-handed neutrinos acquire superheavy masses while breaking lepton number. After the complete symmetry breaking process there is no residual (global) symmetry at all and so the left-handed neutrinos are also obliged to acquire a small (~ eV) mass. The observation of such an effect would be a very low energy signal of lepton number violation. On the other hand grandunified models have been constructed which still retain L_{WS} as a residual global symmetry (Mannheim, 1980, and references therein), so in these models the left-handed neutrinos are still completely massless and the right-handed neutrinos are not superheavy. Thus an experimental study of a possible Majorana mass structure for right- or left-handed neutrinos will eventually provide information about the nature and scale of lepton number violation.

ACKNOWLEDGMENT

This work has been supported in part by the U.S. Department of Energy under grant No. DE-AC02-79ER10336.A.

REFERENCES

Case, K. M. (1957). *Physical Review* **107**, 307.
Frampton, P., Glashow, S. L., and Yildiz, A. (eds.) (1980). *First Workshop on Grandunification* (Math Sci. Press, Brookline, Massachusetts).
Gursey, F. (1958). *Nuovo Cimento* **7**, 411.
Kabir, P. K. (ed.) (1963). *The Development of Weak Interaction Theory* (Gordon and Breach, New York).
Majorana, E. (1932). *Nuovo Cimento* **9**, 335.
Majorana, E. (1937). *Nuovo Cimento* **14**, 170.
Mannheim, P. D. (1979). *Physics Letters* **85B**, 253.
Mannheim, P. D. (1980). *Physical Review D* **22**, 1729.
McLennan, J. A. (1957). *Physical Review* **106**, 821.
Pauli, W. (1957). *Nuovo Cimento* **6**, 204.
Serpe, J. (1952). *Physica* **18**, 295.